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The hidden angular momenta of the coupling–recoupling coefficients of $SU(2)$

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Abstract. A finite projective geometry $T = PG(n, 2)$ is associated with any coupling–recoupling (N-jm) coefficient of $SU(2)$. This geometry is based on a duality of projective spaces and a discrete Fourier transform. An angular momentum j_k with projection M_k is attached at each point $k \in T$. Some of these momenta and projections are specified by the arguments of the N-jm coefficient. The others are qualified as hidden. The value of the N-jm coefficient is given in terms of a summation over the hidden angular momenta and hidden projections of a 'full p_n-JM symbol' with a high degree of symmetry. For the 3-j coefficient (Clebsch–Gordan or Wigner coefficient), the finite projective geometry is a line of three points with one hidden projection and the formula of hidden momenta gives an interpretation of the combinatorial formula of Racah for the 3-j coefficient.

1. Introduction

The angular momentum graphs introduced by Levinson and Yutsis *et al* [16, 25] describe the various coupling–recoupling (N-jm) coefficients of $SU(2)$. In these graphs, momenta are associated with edges and triangular conditions with vertices. Tutte [23] considered the embedding of a general graph in finite projective spaces in connection with the theory of graph colourings. This embedding applied to the angular momentum graphs of $3n$-j coefficients reproduces the geometric description of the $3n$-j coefficient in the finite projective space $P = PG(n, 2)$ that has been considered by Robinson [22]: momenta are associated with points and triangular conditions with collinearity of points. Descriptions of $3n$-j coefficient in real projective spaces have been considered by Fano and Racah ([9], appendices; see also Biedenharn and Louck [2]).

A set of graphical theorems [25] gives practical methods for computing the N-jm coefficients from 3-j and 6-j coefficients. In [14], a combinatorial formula for the N-jm coefficient was derived by a generating function approach based on spaces introduced by Bargmann [1] (for a more recent approach, the chromatic method of evaluating Penrose spin networks, see [12, 13, 17, 18]). Though this formula does not provide an efficient method for computing the N-jm coefficients, it has the interest of being in a combinatorial form that generalizes formulae of Racah [20] for the 3-j and 6-j coefficients. In [15], we introduced, for a $3n$-j coefficient, hidden angular momenta at points of P and a discrete Fourier transform between momenta and 'comomenta' used in the combinatorial formula of the $3n$-j coefficient. Similar Fourier transforms occur in Conway (assisted by Fung) [6] and in Fairlie and Ueno [8]. We then derived the 'formula of hidden momenta' that gives the value of the $3n$-j coefficient in terms of a sum over the hidden momenta of a full 'p_n-J symbol'. For the 6-j coefficient, there is one hidden momentum and the formula is equivalent to the combinatorial formula of Racah ([20], equation (36)).

These results are extended in this paper to any N-jm coefficient of $SU(2)$. We define a finite projective geometry $T = PG(n, 2)$, with hidden angular momenta and hidden projection momenta. The value of the N-jm coefficient is given by a formula of hidden momenta. For the 3-j coefficient, this formula is algebraically equivalent to the formula of Racah ([20], equation (16))

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta_{m_1+m_2+m_3,0} N (-1)^{j_1-j_2-m_3} \sum_z \frac{(-1)^z}{(j_3-j_1-m_2+z)!} \times \frac{1}{(j_3-j_2+m_1+z)!z!(j_2+m_2-z)!(j_1-m_1-z)!(j_1+j_2-j_3-z)!} \tag{1}$$

where z runs over values such that all factorials have arguments in $\mathbb{N}$ (we use the notation $\mathbb{N}$ for the set of natural integers $\{0, 1, 2, \ldots\}$) and where N is a normalizing factor that will be given below in equation (16). Equation (1) becomes the formula of hidden momenta when the sum over z is changed to a sum over M_1 by putting

$$z = [(j_1 - M_1) + (j_2 - M_2) - (j_3 - M_3)]/2 \tag{2}$$

and

$$m_1 = M_2 - M_3 \qquad m_2 = -M_2 \qquad m_3 = M_3. \tag{3}$$

The arguments of the factorials in equation (1) are transformed into

$$\begin{aligned} \alpha_1 &= [-(j_1 + M_1) + (j_2 + M_2) + (j_3 + M_3)]/2 \\ \alpha_2 &= [(j_1 - M_1) - (j_2 - M_2) + (j_3 - M_3)]/2 \\ \alpha_3 &= [(j_1 - M_1) + (j_2 - M_2) - (j_3 - M_3)]/2 \\ \alpha_4 &= [-(j_1 - M_1) + (j_2 - M_2) + (j_3 - M_3)]/2 \\ \alpha_5 &= [(j_1 + M_1) - (j_2 + M_2) + (j_3 + M_3)]/2 \\ \alpha_6 &= [(j_1 + M_1) + (j_2 + M_2) - (j_3 + M_3)]/2 \end{aligned} \tag{4}$$

which we call comomenta of the 3-j coefficient. Equation (4) between comomenta and momenta appears as a discrete Fourier transform. Defining a full 3-JM symbol

$$\left\langle \begin{matrix} j_1 & j_2 & j_3 \\ M_1 & M_2 & M_3 \end{matrix} \right\rangle = \begin{cases} \dfrac{(-1)^{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6}}{\alpha_1!\alpha_2!\alpha_3!\alpha_4!\alpha_5!\alpha_6!} & \text{if} \quad \alpha_i \in \mathbb{N} \quad \text{for} \quad i = 1, 2, \ldots, 6 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

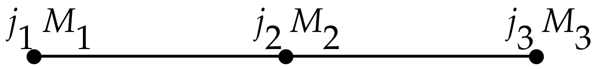
equation (1) reads

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = N \sum_{M_1} (-1)^{\alpha_4+\alpha_5+\alpha_6} \left\langle \begin{matrix} j_1 & j_2 & j_3 \\ M_1 & M_2 & M_3 \end{matrix} \right\rangle. \tag{6}$$

Number M_1 has to take values within the set $\{-j_1, -j_1+1, \ldots, j_1-1, j_1\}$ otherwise the 3-JM symbol of equation (5) is zero. So we interpret M_1 as a projection of j_1 (we say $j_1 M_1$ satisfy *projection conditions*). Equation (6) is the formula of hidden momenta for the 3-j coefficient.

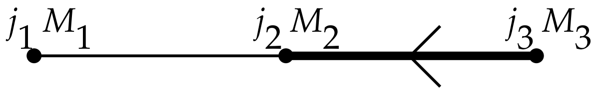
The full 3-JM symbol (5) has a high degree of symmetry. It is invariant in the $6! = 720$ permutations of the comomenta. For the sum in equation (6), there remain the 72 symmetries of the 3-j coefficient (Regge [21]).

Geometrically, the momenta and projections are defined on the *momentum line* $PG(1,2)$ which contains only three points:



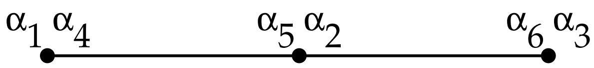
(7)

A momentum j_k and its projection M_k are associated to point $k = 1$, 2 or 3, and the line expresses that $j_1 j_2 j_3$ satisfy triangular conditions. There is only one hidden projection (we can take any one of M_1, M_2 or M_3). Equation (3) is represented geometrically by the M-chain $(3,2)$ consisting of two ordered points, 3 and 2, and of one line 32:



(8)

The comomenta are defined on a dual *comomentum line*:



(9)

with a pair of comomenta at each point.

Here is the plan of our exposition. The general N-jm coefficient is defined from its angular momentum graph G (section 2). We then review the diagrams drawn on G (section 3), the combinatorial formula (section 4) and the projective geometry of the $3n$-j coefficient (section 5). We have then the elements and notations to construct the projective geometry of the N-jm coefficient (section 6).

2. Angular momentum graphs

The *angular momentum graphs* represent the N-jm coefficients. Various versions of these graphs have been considered [5, 7, 11, 16, 24, 25]. We shall use the following simple variant. The 3-j coefficient $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ is represented by figure 1 which serves as the basic building block. The sign $(\pm)$ at the vertex indicates the cyclic order of the momenta in the 3-j symbol. An angular momentum graph is obtained from these building blocks by a sequence of *contractions*. A contraction corresponds to a summation of the form

$$\sum_m (-1)^{j-m} \begin{pmatrix} \ldots j \ldots \\ \ldots m \ldots \end{pmatrix} \begin{pmatrix} \ldots j \ldots \\ \ldots -m \ldots \end{pmatrix}. \tag{10}$$

Letting L and R be the graphs of the 3-j coefficients $\begin{pmatrix} \ldots j \ldots \\ \ldots m \ldots \end{pmatrix}$ and $\begin{pmatrix} \ldots j \ldots \\ \ldots -m \ldots \end{pmatrix}$ respectively, the contraction is represented by joining the edges j of L and R by an arrow going from L to R. For example the 4-jm coupling coefficient

$$\begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}_{j_5} = \sum_{m_5} (-1)^{j_5 - m_5} \begin{pmatrix} j_1 & j_2 & j_5 \\ m_1 & m_2 & m_5 \end{pmatrix} \begin{pmatrix} j_3 & j_4 & j_5 \\ m_3 & m_4 & -m_5 \end{pmatrix} \tag{11}$$

is represented by figure 2.

The most general graph G so obtained is a trivalent graph (three edges meet at each of the v vertices) with h components, f *free edges* linked to one vertex only and g *bound edges* linked to vertices at each extremity (when the extremities are linked to the same vertex, the edge is a

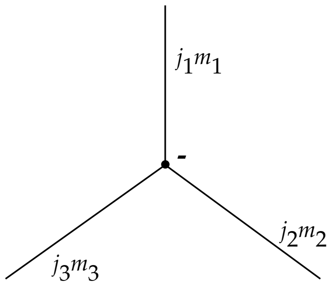


Figure 1. The 3-j coefficient.

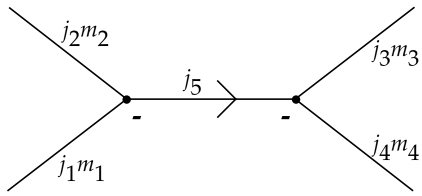


Figure 2. A 4-jm coupling coefficient.

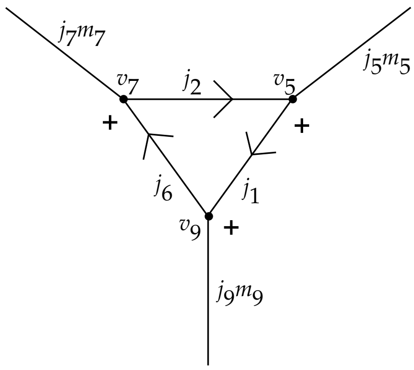


Figure 3. A 3-jm coupling–recoupling coefficient.

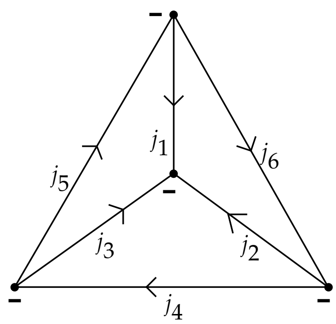


Figure 4. The 6-j coefficient $\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$.

loop). Each edge is labelled with an angular momentum j accompanied by its projection m in the case of a free edge. The graph is decorated with arrows on bound edges (changing the direction of an edge j multiplies the coefficient by $(-1)^{2j}$) and $\pm$ signs at vertices (changing the sign of a vertex where $j_1 j_2 j_3$ meet multiplies the coefficient by $(-1)^{j_1+j_2+j_3}$). We are considering only coefficients whose projections add up to zero ($\sum m = 0$), so that we do not decorate the free edges of the graphs. Usually, we are interested in connected graphs ($h = 1$). Coupling coefficients are then represented by *trees*, as in figure 2. We have a coupling–recoupling coefficient as in figure 3 when the graph has *circuits* and a $3n$-j coefficient when the graph has no free edge as in figure 4. We use the name N-jm coefficient for the most general graph (including $3n$-j) and 3-jm (4-jm) coefficients for graphs with 3 (4) free edges.

3. The open and closed diagrams

In this section, we define subsets of edges of G that are used to express the value of the N-jm coefficient. The following presentation is an adaptation to our needs of the definition of cycles by Tutte [23] (see also Holton and Sheenhan [10]).

Let E be the set of edges of graph G. For figure 3, $E = \{1, 2, 5, 6, 7, 9\}$ where we designate the edges by integers. The *edge space* $\mathcal{E}$ of the graph is the vector space over the 2-element field $\mathbb{F}_2 = \{0, 1\}$ of functions $E \to \mathbb{F}_2$. The support of a function $w \in \mathcal{E}$ is the subset $W \subseteq E$ of the edges $e \in E$ such that $w(e) = 1$. We shall not distinguish between a function and its support. The sum of two edge subsets $W, W' \subseteq E$ is then their symmetric

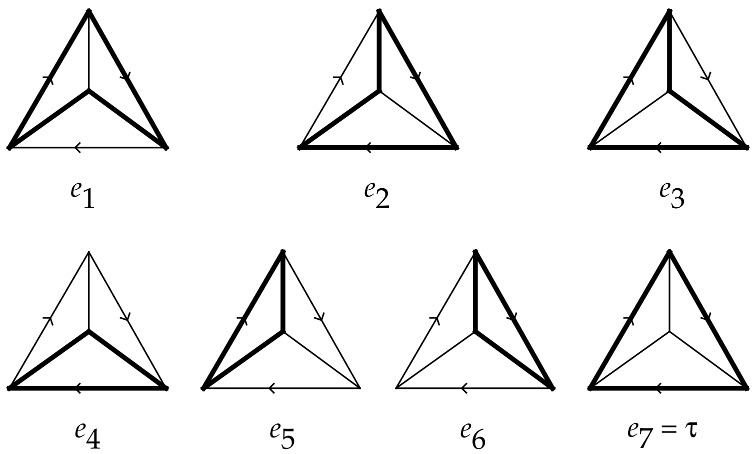


Figure 5. The closed diagrams of the 6-j coefficient.

difference $(W \cup W') \setminus (W \cap W')$.

Let V be the set of vertices of graph G. For figure 3, $V = \{v_5, v_7, v_9\}$ where v_k is the vertex incident with free edge k. We associate with each free edge a supplementary 1-valent vertex placed at the unlinked extremity of the edge and called its *end*. We obtain in this way a graph G' with $v + f$ vertices and $g + f$ edges. Its edge space can be identified with the edge space $\mathcal{E}$ of G. The *vertex space* $\mathcal{V}$ is the vector space over $\mathbb{F}_2$ of functions $V' \to \mathbb{F}_2$ from the set of vertices V' of G' to $\mathbb{F}_2$. Similarly as for the edge space, we do not distinguish between a function $\phi \in \mathcal{V}$ and the subset of V' which is the support of ϕ.

Let $a \in E$ be an edge in G or G' and $s, s' \in V'$ be the *endvertices* of a, that is the vertices in G' incident with a. We have $s = s'$ when a is a loop. Letting $w_a \in \mathcal{E}$, $\phi_s \in \mathcal{V}$, $\phi_{s'} \in \mathcal{V}$ correspond respectively to the subsets $\{a\} \subseteq E$, $\{s\}, \{s'\} \subseteq V'$, we can define a linear function $\mathcal{E} \xrightarrow{\partial} \mathcal{V}$ such that

$$\partial w_a = \phi_s + \phi_{s'}. \tag{12}$$

This function ∂ gives the endvertices of an edge, or, more generally, of a subset of edges. For example, if the endvertices of edge b are s' and s'', the subset $\{a, b\} \subseteq E$ corresponding to $w_{ab} = w_a + w_b \in \mathcal{E}$ form a path with two extremities s and s'' given by

$$\partial w_{ab} = \partial w_a + \partial w_b = (\phi_s + \phi_{s'}) + (\phi_{s'} + \phi_{s''}) = \phi_s + \phi_{s''}. \tag{13}$$

A *cycle* of graph G is a subset of edges $w \in \mathcal{E}$ such that $\partial w = 0$. The connected components of a cycle are called *circuits*. In other words, a cycle is a subset of edges that is a union of any number of disjoint circuits. For the 6-j coefficient, there are $8 = 2^3$ cycles, the seven cycles represented in figure 5 and the empty set (the *zero cycle*). Note that we have changed the labelling of [15] for these cycles. Given two cycles e_1, e_2 and scalars $\lambda_1, \lambda_2 \in \mathbb{F}_2$, the linear combination $\lambda_1 e_1 + \lambda_2 e_2$ is a cycle: the cycles form the *cycle-subspace* $\mathcal{C}$ of $\mathcal{E}$. The dimension of $\mathcal{C}$ for graph G is $g - v + h$ ([3, 4]). The non-zero cycles, also called *closed diagrams*, form the set $\mathcal{C} \setminus \{0\}$ of $q = 2^{g-v+h} - 1$ elements.

For two different free edges a and b, with respective ends s_a and s_b (that we identify with elements of $\mathcal{V}$), an *open diagram of type* $a \to b$ is a subset of edges $\omega \in \mathcal{E}$, such that $\partial\omega = s_a + s_b$. We distinguish open diagrams of types $a \to b$ and $b \to a$ corresponding to the same $\omega \in \mathcal{E}$, and call them *reversed diagrams*. In other words, we can think of an open diagram of type $a \to b$ as a subset of edges formed of the disjoint union of one oriented path

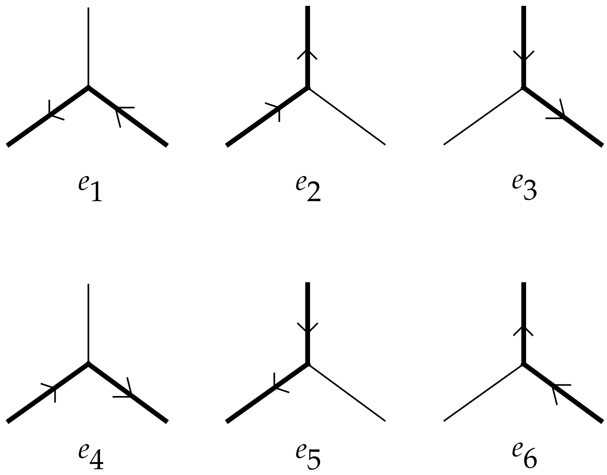


Figure 6. The open diagrams of the 3-j coefficient.

going from edge a to b, that we call *open path*, and any number of circuits. We denote by Ω_{ab} the set of open diagrams of type $a \to b$. If a and b are in different components of graph G, there is no open diagram of type $a \to b$ and Ω_{ab} is empty. Otherwise, if $\omega, \omega' \in \Omega_{ab}$, then $\omega - \omega'$ is a cycle and Ω_{ab} is an affine subspace of $\mathcal{E}$ containing 2^{g-v+h} elements. The set of all open diagrams Ω contains, in the case of a connected graph, $p = f(f-1)2^{g-v+1}$ elements. For the 3-j coefficient, there are six open diagrams, represented in figure 6 and denoted by e_1, $e_2, \ldots, e_6$. There are three pairs of reversed diagrams: (e_1, e_4), (e_2, e_5) and (e_3, e_6).

For a connected $3n$-j coefficient ($v = 2n$, $f = 0$, $g = 3n$, $h = 1$), the dimension of $\mathcal{C}$ is $n + 1$, there are $q = 2^{n+1} - 1$ closed diagrams and no open diagram ($p = 0$).

For a connected coupling coefficient with v couplings ($f = v + 2$, $g = v - 1$, $h = 1$), there are $p = (v + 2)(v + 1)$ open diagrams and no closed diagram ($\dim \mathcal{C} = 0$, $q = 0$).

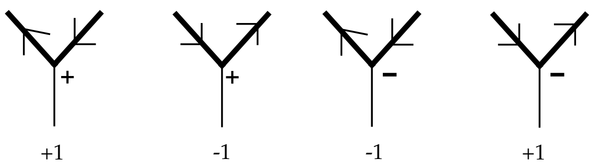
4. The combinatorial formula

In this section we review the combinatorial formula of the N-jm coefficient [14], which is based on the open and closed diagrams of G. We follow a presentation similar to the one we used in [15] for the combinatorial formula of the $3n$-j coefficient. We say that a vertex of G is a vertex of diagram e_i if it is incident with edges of e_i. To each diagram e_i is associated a *sign* $\epsilon_i = \pm 1$ computed by the following rules:

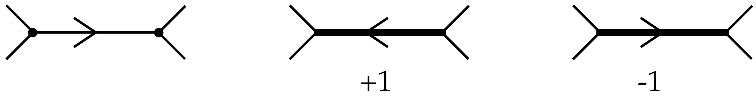
(i) Orient all circuits of e_i in an arbitrary fashion.

(ii) Multiply the factors:

- at each vertex of e_i, a factor of +1 if the order of the edges is (incoming edge, outgoing edge, third edge) and -1 otherwise:



- on each bound edge of e_i, a factor of +1 if the directions of the edge and diagram are opposite or -1 if they are the same:



- by a factor of -1 for each circuit.

There is an even (resp. odd) total number of edges and vertices on a circuit (resp. open path). The sign of a diagram is thus independent of the orientations chosen on the circuits. The signs of reversed open diagrams are opposite.

Examples. For the 6-j coefficient (figures 4 and 5) $\epsilon_i = 1$ for all closed diagrams; for the 3-j coefficient (figures 1 and 6) $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$ and $\epsilon_4 = \epsilon_5 = \epsilon_6 = -1$.

To each set of values of momenta in a coupling–recoupling coefficient we associate an array x of these values. For example, we associate the 2×3 array $x = \begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{bmatrix}$ to the 3-j coefficient $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$. We denote by R the space of arrays like these when the entries are integers or half-integers that satisfy the triangular and projection conditions of the N-jm coefficient. We denote by $\{x\}$ the value of the coefficient associated to array $x \in R$.

For arrays in R, we have the usual addition and multiplication by a scalar $\lambda \in \mathbb{N}$. For example, in the case of the 3-j coefficient, if $x' = \begin{bmatrix} j'_1 & j'_2 & j'_3 \\ m'_1 & m'_2 & m'_3 \end{bmatrix}$ we have $x + x' = \begin{bmatrix} j_1 + j'_1 & j_2 + j'_2 & j_3 + j'_3 \\ m_1 + m'_1 & m_2 + m'_2 & m_3 + m'_3 \end{bmatrix}$ and $\lambda x = \begin{bmatrix} \lambda j_1 & \lambda j_2 & \lambda j_3 \\ \lambda m_1 & \lambda m_2 & \lambda m_3 \end{bmatrix}$. It is easy to see that if $x, x' \in R$, $\lambda \in \mathbb{N}$ then $x + x' \in R$, $\lambda x \in R$ (R is closed under addition and multiplication by a non-negative integer scalar).

To each diagram e_i we associate an array in R corresponding to momenta of $\frac{1}{2}$ on the edges of e_i, and, if $e_i \in \Omega_{ab}$, with projection $\frac{1}{2}$ (resp. $-\frac{1}{2}$) on free edge b (resp. a). The remaining momenta and projections are zero. To simplify notations, these elements of R are denoted by the same name as the diagrams. They are for the 3-j coefficient:

$$\begin{aligned} e_1 &= \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} & e_2 &= \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} & e_3 &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \\ e_4 &= \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} & e_5 &= \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} & e_6 &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}. \end{aligned} \tag{14}$$

Each element $x \in R$ can be decomposed over these $p + q$ arrays $e_i \in R$ ($i = 1, 2, \ldots, p$ corresponds to the p open diagrams and $i = p + 1, p + 2, \ldots, p + q$ corresponds to the q closed diagrams) as

$$x = \sum_{i=1}^{p} \alpha_i e_i + \sum_{i=p+1}^{p+q} \beta_i e_i \qquad \alpha_i, \beta_i \in \mathbb{N}. \tag{15}$$

We call α_i (resp. β_i) the [α- (resp. β-)] *comomentum* associated to e_i. In case p or q is zero, the corresponding sum is omitted in equation (15). Since the arrays e_i are not independent in R, decomposition (15) is not unique in general, but the number of different decompositions is always finite because the comomenta have to be non-negative integers.

The normalizing factor N of the coefficient is the product of the triangle factors Δ_{abc} of the v couplings (a, b, c) (one for each vertex) and of the f free edge factors N_{jm} (one for each free edge):

$$N = \prod_{(a,b,c)} \Delta_{abc} \prod_{(j,m)} N_{jm} \tag{16}$$

$$\Delta_{abc} = \left(\frac{(a+b-c)!(b+c-a)!(c+a-b)!}{(a+b+c+1)!} \right)^{1/2} \tag{17}$$

$$N_{jm} = ((j+m)!(j-m)!)^{1/2}. \tag{18}$$

The value of the coefficient is expressed as (see [14], equation (16))

$$\{x\} = N \sum \frac{(|\alpha|+|\beta|+1)!}{(|\alpha|+1)!} \prod_{i=1}^{p} \frac{(-\epsilon_i)^{\alpha_i}}{\alpha_i!} \prod_{i=p+1}^{p+q} \frac{(-\epsilon_i)^{\beta_i}}{\beta_i!} \tag{19}$$

where the sum is over the decompositions (15) of x in comomenta and where $|\alpha| = \sum_{i=1}^{p} \alpha_i$, $|\beta| = \sum_{i=p+1}^{p+q} \beta_i$.

Equation (19) is a K-fold summation, where

$$K = p + q - I \tag{20}$$

is the difference between the number $p+q$ of comomenta and the number I of independent momenta and projections. For a connected $3n$-j coefficient ($I = g = 3n$, $p = 0$, $q = 2^{n+1} - 1$) $K = 2^{n+1} - 1 - 3n$. For a connected coupling coefficient with v couplings ($p = (v+2)(v+1)$, $q = 0$), since the projections have sum 0, $K = p - (g + 2f - 1) = v^2$.

Example (3-j coefficient). In the case of the 3-j coefficient, decomposition (15) $x = \sum_{i=1}^{6} \alpha_i e_i$, where the $e_i \in R$ are given by equation (14), expresses the momenta and projections of the 3-j coefficient corresponding to a set of values of the comomenta $(\alpha_i)_{1 \leqslant i \leqslant 6}$ as

$$\begin{aligned}
j_1 &= (+ \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6)/2 \\
j_2 &= (+\alpha_1 + \alpha_3 + \alpha_4 + \alpha_6)/2 \\
j_3 &= (+\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5)/2 \\
m_1 &= (+ \alpha_2 - \alpha_3 - \alpha_5 + \alpha_6)/2 \\
m_2 &= (-\alpha_1 + \alpha_3 + \alpha_4 - \alpha_6)/2 \\
m_3 &= (+\alpha_1 - \alpha_2 - \alpha_4 + \alpha_5)/2.
\end{aligned} \tag{21}$$

The value of the 3-j coefficient, equation (19), reads

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = N \sum \frac{(-1)^{\alpha_1+\alpha_2+\alpha_3}}{\alpha_1!\alpha_2!\alpha_3!\alpha_4!\alpha_5!\alpha_6!} \tag{22}$$

where $N = \Delta_{j_1 j_2 j_3} N_{j_1 m_1} N_{j_2 m_2} N_{j_3 m_3}$ and where the sum is over the sets of comomenta with values in $\mathbb{N}$ satisfying the set of equations (21). Solving equations (21) for the comomenta α_i in terms of j_1, j_2, j_3, m_1, m_2 and $z = \alpha_3$ and using these expressions in equation (22) gives Racah's formula (1).

5. The projective geometry of the $3n$-j coefficient

In this section we review results of [15] for the $3n$-j coefficient. We assume for simplicity graph G to be connected and without cuts on one or two edges. The projective space $P^* = PG(n^*, 2)$

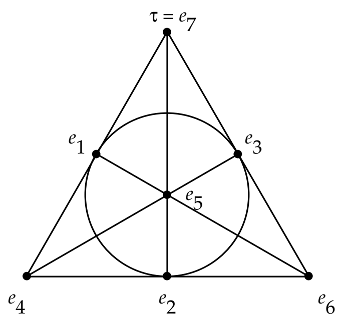


Figure 7. The Fano plane of comomenta P^* for the 6-j coefficient.

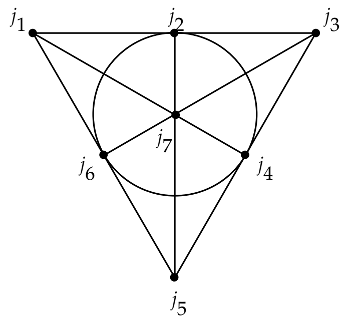


Figure 8. The Fano plane of momenta P for the 6-j coefficient.

is identified with $\mathcal{C} \setminus \{0\}$, the set of the $p_n = 2^{n+1} - 1$ closed diagrams of the $3n$-j coefficient. In the dual projective space $P = PG(n, 2)$, $3n$ points are identified with edges of G by

$$\begin{array}{l}\text{edge } k \in G \text{ is identified with point } k \in P \text{ such that the}\\ 2^n \text{ cycles that contain edge } k \text{ are the points of } P^* \setminus k^*\end{array} \tag{23}$$

where we denote by $k^* \subset P^*$ the dual hyperplane of k. We also denote by $E \subset P$ the set of the points identified as edges of G. We call E the embedding of G in P. When three edges of G are incident at one vertex, the corresponding points in P are collinear. By duality of property (23) we have

$$\text{the set of edges of cycle } i \in P^* \text{ is } E \setminus i^* \tag{24}$$

where $i^* \subset P$ is the hyperplane dual to i. Hidden momenta are associated with the $p_n - 3n$ points of $P \setminus E$. For the 6-j coefficient these projective spaces are Fano planes (figures 7 and 8) with one hidden momentum j_7.

For each point $k \in P$, we denote by j_k the associated visible (already in G) or hidden momentum and by χ_k the irreducible character of the Abelian group $\mathcal{C}$ defined by

$$\chi_k(i) = \begin{cases} 1 & \text{if} \quad i \in k^* \quad \text{or} \quad i = 0 \\ -1 & \text{otherwise.} \end{cases} \tag{25}$$

The comomenta l_i associated to points $i \in P^*$ are expressed in terms of momenta as a discrete Fourier transform

$$l_i = -\frac{1}{2^{n-1}} \sum_{k \in P} \chi_k(i) j_k \qquad \text{for} \quad i \in P^* \tag{26}$$

with inverse transform

$$j_k = \tfrac{1}{2} \sum_{i \in P^* \setminus k^*} l_i \qquad \text{for} \quad k \in P. \tag{27}$$

Let us say that when we give to the β-comomenta *specified values*, matrix equation (15) (here $p = 0$) determines *sample values* of the momenta j_k. The sample value of j_k for $k \in E \subset P$ is also given by equation (27), with l_i identified with the β-comomentum associated to cycle i. The set of equations (27) is thus an enlargement of matrix equation (15) and, since it is invertible, we can express equation (19) as follows.

Denoting by X an array of p_n angular momenta j_k ($k \in P$), the full p_n-J symbol $\langle X \rangle$ is defined in terms of the comomenta (26) by

$$\langle X \rangle = \begin{cases} \dfrac{(-1)^{|l|}(|l|+1)!}{\prod_{i\in P^*} l_i!} & \text{if} \quad \forall i \in P^* \quad l_i \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \tag{28}$$

where $|l| = \sum_{i\in P^*} l_i$. The value of the $3n$-j coefficient is given by the formula of hidden momenta,

$$\{x\} = N \sum (-1)^{t(X)} \langle X \rangle \qquad (-1)^{t(X)} = \prod_{i\in P^*} (\epsilon_i)^{l_i} \tag{29}$$

where N is as in equation (16) and where the sum is over the hidden momenta of the full p_n-J symbol. The condition $\forall i \in P^*,\ l_i \in \mathbb{N}$ in equation (28) implies that the full p_n-J symbol is zero if there is a line abc in P such that $j_a j_b j_c$ do not satisfy triangular conditions. The sum in equation (29) is thus limited by the triangular conditions associated to the $p_n(p_n-1)/6$ lines of the projective space.

6. The projective geometry of the N-jm coefficient

In this section we generalize the results of the preceding section to any N-jm coefficient with graph G. The N-jm coefficient is still described by a finite projective space (noted T), but at each point $k \in T$ are attached a momentum j_k with projection M_k, and at each point in the dual space a pair of comomenta. The geometry of the N-jm coefficient depends not only on the embedding $E \subset T$ of G, but also on a M-chain that specifies the projections m_a of the N-jm coefficient in terms of the M_k of T. We present a construction of this geometry based on a $3n$-j coefficient obtained by completing graph G.

6.1. The graph $\overline{G}$

Let us denote the f free edges of G by numbers $1, 2, \ldots, f$ (with $i \equiv i + f$) and their ends by s_i, thus fixing an (arbitrary) cyclic order. By adding f *outer edges* $s_i s_{i+1}$, labelled with τ_i ($i \equiv 1, 2, \ldots, f$) to graph G, we construct a trivalent graph $\overline{G}$ that we call a *completion* of G. Putting $n = (g + 2f - 3)/3$, $\overline{G}$ is the graph of a $3(n+1)$-j coefficient with $3(n+1)$ edges and $2(n+1)$ vertices. We call the cycle $\tau = \{\tau_1, \tau_2, \ldots, \tau_f\}$ of $\overline{G}$ the *outer cycle*.

Example (3-j coefficient). We take the cyclic order 123 for the free edges of G = figure 1. We have $n = 1$ and $\overline{G}$ is figure 4 of the 6-j coefficient with outer cycle $\tau = \{4, 5, 6\}$.

For each open diagram $\omega \in \Omega_{ab}$ of G we define a closed diagram of $\overline{G}$, noted $\overline{\omega}$ and called the *completion* of ω, obtained by adjoining to ω the outer edges $\tau_b, \tau_{b+1}, \ldots, \tau_{a-1}$. Graphically, to obtain $\overline{\omega}$ from ω, we join b to a on the outer cycle going in the cyclic order. For the 3-j coefficient, the completion of the open diagram e_i ($i = 1, 2, \ldots, 6$) in figure 6 is the closed diagram e_i of the 6-j coefficient in figure 5. We say that cycles of $\overline{G}$ of the form $\overline{\omega}$ are of *type* α. Note that $\overline{\omega}' = \overline{\omega} + \tau$ is also of type α and is the completion of the open diagram $\omega' \in \Omega_{ba}$ reversed of $\omega \in \Omega_{ab}$.

We define the completion $\overline{w}$ of a closed diagram w of G to be w itself when considered as a cycle of $\overline{G}$. Such a cycle $\overline{w}$ of $\overline{G}$ is said of *type* β, and cycle $\overline{w} + \tau$ of $\overline{G}$ is said of *type* γ. The closed diagrams of $\overline{G}$ are thus classified in types τ (for the outer cycle), α, β, γ and δ (for the remaining diagrams).

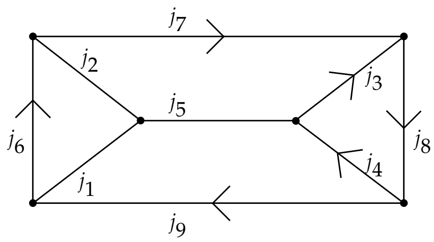


Figure 9. Completion of figure 2 or 3.

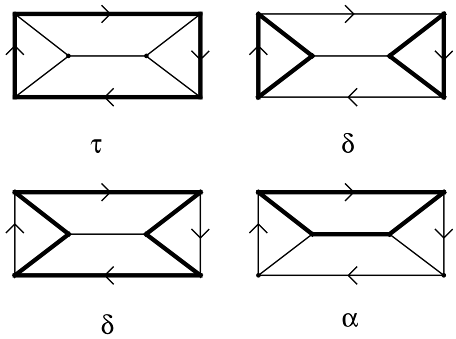


Figure 10. Closed diagrams of types τ, δ and α in the completion of figure 2. Arrows show the cyclic order on the outer cycle.

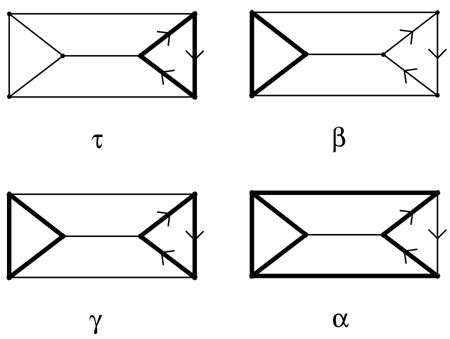


Figure 11. Closed diagrams of types τ, β, γ and α in the completion of figure 3.

Examples. Taking figure 9 as a completion of figure 2, figure 10 shows the outer cycle τ, the two diagrams of type δ, and one diagram of type α which is the completion of an open diagram of Ω_{32}; the remaining 11 diagrams of $\overline{G}$ are of type α. Taking the same figure 9 as a completion of figure 3, figure 11 shows the outer cycle τ, the only diagrams of types β and γ, and one diagram of type α which is the completion of an open diagram of Ω_{79}; the remaining 11 diagrams of $\overline{G}$ are of type α.

We define the projective spaces P and P^* for $\overline{G}$ as in section 5, but their dimension is now $n+1$ instead of n and the number of points of P (or P^*) is $p_{n+1} = 2p_n + 1$ with $p_n = 2^{n+1} - 1$.

6.2. The projective space T of momenta and the M-chain F

Let T be the hyperplane of P dual to the outer cycle $\tau \in P^*$. By property (24), T contains all edges of G. We call T the *projective space of momenta* of G, and say as before that the set of edges E is embedded in T. This embedding $E \subset T$ is the same as the embedding of $\widehat{G}$ in $PG(n, 2)$, where graph $\widehat{G}$ is obtained by joining all f ends of G (or shrinking cycle τ of $\overline{G}$ to a point). It is thus independent of the cyclic order of the free edges used to construct P.

The outer edges $\tau_1, \ldots, \tau_f$ of $\overline{G}$ are identified with points of $P\backslash T$. Choosing any $t \in P\backslash T$, that we call the *centre* of P, we put, if $t \neq \tau_a$, $\rho_a = t + \tau_a \in T$ for $a = 1, 2, \ldots, f$. There are two kinds of choices of the centre t:

- The centre is different from all outer edges. We call the ordered set of the f points ρ_a and f lines $\rho_{a-1}\rho_a$ ($a \equiv 1, 2, \ldots, f$) the *(closed) M-chain* $F = (\rho_1, \rho_2, \ldots, \rho_f)$.
- The centre is one of the outer edges. Let us take $t = \rho_f$. The above F exists in $T \cup \{0\}$ with $\rho_f = 0$. By keeping only its points and lines in T, we obtain the *(open) M-chain* $F = (\rho_1, \rho_2, \ldots, \rho_{f-1})$ of $f - 1$ points and $f - 2$ lines.

Free edge a of G is incident with the outer edges τ_{a-1} and τ_a, so that $a = \rho_{a-1} + \rho_a$, which means that the M-chain has the following property:

free edge a of G is the third point on line $\rho_{a-1}\rho_a$ of the M-chain. For an open M-chain, the beginning ρ_1 (resp. ending ρ_{f-1}) of the M-chain is identified with free edge 1 (resp. f) of G. (30)

Example (3-j coefficient). The first case occurs when we take t at j_7. The closed M-chain is $(1, 2, 3)$ with free edge 1 (resp. 2, 3) being the third point on line 23 (resp. 31, 12). The second case occurs for the other choices of the centre. For t at j_4, the open M-chain $(3, 2)$ is shown in equation (8).

6.3. The projective space T^* of comomenta

Let T^* be the hyperplane of P^* dual to the centre $t \in P$. Considering T and T^* as dual projective spaces $PG(n, 2)$ of dimension n, we call T^* the *projective space of comomenta* of G. At point $i \in T^*$, we have already comomentum l_i of P^*, which we call *upper comomentum* of T^*. We also associate to $i \in T^*$ the *lower comomentum* $l'_i = l_{\tau+i}$, which is the comomentum in P^* at the third point on line τi. The types (α, β, γ or δ) of these upper and lower comomenta are the types of the corresponding cycles in P^*.

At point $k \in T$, we already have momentum j_k of P. We also associate to k a projection

$$M_k = j_{t+k} - j_t \tag{31}$$

which is the difference of the momenta at point $r = t + k$ in $P \setminus T$ (the third point on the line joining k to the centre) and at the centre. The triangular conditions $j_k j_r j_t$ imply that $j_k M_k$ satisfies projection conditions.

For $k \in T$, the discrete Fourier transform expressing the momenta in terms of comomenta is (see the appendix):

$$j_k = \tfrac{1}{2} \sum_{i \in T^* \setminus k^*} (l_i + l'_i) \tag{32}$$

$$M_k = \tfrac{1}{2} \sum_{i \in T^* \setminus k^*} (l_i - l'_i) \tag{33}$$

with inverse transform for $i \in T^*$:

$$l_i = -\frac{1}{2^n} \sum_{k \in T} \chi_k(i)(j_k + M_k) \tag{34}$$

$$l'_i = -\frac{1}{2^n} \sum_{k \in T} \chi_k(i)(j_k - M_k). \tag{35}$$

Let us now show that we can identify which comomenta $l_i l'_i$ of types α and β are associated to closed and open diagrams of G only from the embedding $E \subset T$ and the M-chain F. Using property (24), we obtain the following, which proves also that the $2^{n+1}(f-1)!$ arbitrary choices (cyclic ordering of the f free edges, choice of the centre among 2^{n+1} points) in the construction of T and T^* are completely encoded by the M-chain F:

- If $i = \overline{w} \in T^*$ is the completion of a closed diagram w of G, then $E \setminus i^*$ is the set of edges of w. The upper comomentum l_i is then associated to w and the lower comomentum l'_i is of type γ. Note that, since the completion of any closed diagram of G is in T^*, comomenta of type β are always upper comomenta.

- If $i = \overline{\omega} \in T^*$ is the completion of an open diagram $\omega \in \Omega_{ab}$ of G, then $E \setminus i^*$ is the set of edges of ω and $F \setminus i^*$ is the open chain $\rho_b \rho_{b+1} \ldots \rho_{a-1}$. Comomentum l_i is then associated to ω and l'_i to the reversed open diagram ω'.

We also denote the comomenta of types α and β by $(\alpha_i)_{1 \leqslant i \leqslant p}$ and $(\beta_i)_{p<i \leqslant p+q}$, labelled accordingly to the above associations, and those of types γ and δ by $(\gamma_i)_{p+q<i \leqslant p+2q}$ and $(\delta_i)_{p+2q<i \leqslant 2p_n}$, labelled arbitrarily ($\alpha\beta\gamma\delta$ *notation* for comomenta).

Example (3-j coefficient). When we take the centre t of P at j_4, T^* is the comomentum line $e_1 e_5 e_6$. The ordered pair of comomenta $l_i l'_i$ at e_1 (resp. e_5, e_6) is the pair $\alpha_1 \alpha_4$ (resp. $\alpha_5 \alpha_2$, $\alpha_6 \alpha_3$) as pictured in equation (9). Equations (34) and (35) give equation (4).

6.4. The formula of hidden momenta

Let us give specified values to $(\alpha_i)_{1 \leqslant i \leqslant p}$ and $(\beta_i)_{p<i \leqslant p+q}$ and so to comomenta of P^* of types α and β. We call, as before, *sample values* the values of momenta j_k and projections m_a of the N-jm coefficient that result from matrix equation (15). We put $l_i = 0$ for the comomenta of P^* of types γ and δ. All upper and lower comomenta of T^* have then specified values. We use these comomenta of P^* (with an arbitrary value for comomentum l_τ at the outer cycle) in equation (27) to compute the momenta at edges of $\overline{G}$. The sample value of j_k at edge k of G is the same as momentum at edge k of $\overline{G}$ and the sample value of m_a on free edge a of G is given by $j_{\tau_a} - j_{\tau_{a-1}}$, the difference of momenta on the outer edges τ_a and τ_{a-1} of $\overline{G}$ adjacent to a. The sample value of j_k at edge k of G is thus given by equation (32). The sample value of m_a is related to the projections M_k (defined by equation (31) and given by equation (33)) at points of the M-chain. If the M-chain is closed,

$$m_a = M_c - M_b \tag{36}$$

where $b = \rho_{a-1}$ and $c = \rho_a$ are collinear with a by property (30). If the M-chain is open, with beginning $d = \rho_1$ and ending $e = \rho_{f-1}$, equation (36) is replaced by

$$m_d = M_d \qquad m_e = -M_e \tag{37}$$

for free edges d and e in G.

Example (3-j coefficient). When we take the centre t of P at j_7, the projections M_k are defined from

$$M_1 = j_4 - j_7 \qquad M_2 = j_5 - j_7 \qquad M_3 = j_6 - j_7 \tag{38}$$

and the closed M-chain $(1, 2, 3)$ corresponds to relations

$$m_1 = M_3 - M_2 \qquad m_2 = M_1 - M_3 \qquad m_3 = M_2 - M_1. \tag{39}$$

When we take the centre t of P at j_4, the projections M_k are defined from

$$M_1 = j_7 - j_4 \qquad M_2 = j_6 - j_4 \qquad M_3 = j_5 - j_4 \tag{40}$$

and the open M-chain $(3, 2)$ corresponds to relations (3).

The system of equations (32) and (33) is thus an enlargement of the matrix equation (15). It defines an array X of $2p_n$ values of momenta and projections $(j_k M_k)_{k \in T}$ associated with the p_n points of the projective space T for a specified set of comomenta. We call *hidden momenta* the j_k that do not correspond to edges in G. The projections M_k on the M-chain are related to the projections m_a of the N-jm coefficient by equations (36) and (37). In the case of an open

M-chain, these $f-1$ projections are completely determined (they are *visible projections*). In the case of a closed M-chain, these f projections form a set with one *hidden projection* (any one of them). The remaining M_k, at points not in the M-chain, are also called *hidden projections*. It can surprisingly happen that at free edge k (where $j_k m_k$ are known) projection M_k is hidden.

Example (3-j coefficient). When we take the centre t of P at j_4, the open M-chain is $(3, 2)$. There is only one hidden projection M_1. The system of equations (32) and (33) is an enlargement of equation (21) (with $M_2 = -m_2$ and $M_3 = m_3$) containing the additional equation for hidden projection M_1:

$$M_1 = (-\alpha_2 - \alpha_3 + \alpha_5 + \alpha_6)/2. \tag{41}$$

The inverse of the system of equations (21) and (41) is the system of equations (4) (which are the same as equations (34) and (35)).

For arbitrary values of momenta j_k and projections M_k in array X, we calculate the $2p_n$ comomenta of T^* by equations (34) and (35). The full p_n-JM symbol $\langle X\rangle$ is defined by, using the $\alpha\beta\gamma\delta$ notation for comomenta,

$$\langle X\rangle = \begin{cases} \dfrac{(|\alpha|+|\beta|+1)!}{(|\alpha|+1)!} \dfrac{(-1)^{|\alpha|+|\beta|}}{\prod_{i=1}^{p}\alpha_i!\prod_{i=p+1}^{p+q}\beta_i!} & \begin{array}{l}\text{if all} \quad \alpha_i, \beta_i \in \mathbb{N} \\ \text{and all} \quad \gamma_i = 0, \delta_i = 0\end{array} \\ 0 & \text{otherwise.} \end{cases} \tag{42}$$

We rewrite equation (19) as the *formula of hidden momenta*: the value $\{x\}$ of the N-jm coefficient is

$$\{x\} = N\sum(-1)^{t(X)}\langle X\rangle \qquad (-1)^{t(X)} = \prod_{i=1}^{p}(\epsilon_i)^{\alpha_i}\prod_{i=p+1}^{p+q}(\epsilon_i)^{\beta_i} \tag{43}$$

where N is given by equation (16) and where the sum is over the hidden momenta and hidden projections of the full p_n-JM symbol.

The sum in equation (43) is limited by triangular conditions $j_a j_b j_c$ (for each line abc in T), as in equation (29) for the $3n$-j coefficient, and by projection conditions $j_k M_k$ (for each point $k \in T$). The conditions $\gamma_i = 0$, $\delta_i = 0$ in equation (42) have the effect of imposing $2p_n - p - q$ relations between hidden momenta and projections. If we want to determine a set of independent hidden momenta and projections, for each pair of conditions $l_i = l_i' = 0$ at a cycle $i \in T^*$ of type δ, we remove one hidden momentum and one hidden projection and for each condition $\gamma_i = 0$, we remove one hidden projection. The total number K of independent hidden momenta and projections is given by equation (20).

In the case of a $3n$-j coefficient, the above construction remains valid when we take for the outer cycle τ added to G a loop disconnected from G. All comomenta come in pairs of type $\beta\gamma$ so that conditions $\gamma_i = 0$ impose $M_k = j_k$ for all projections. The simpler geometry of the $3n$-j coefficient is recovered by ignoring the γ comomenta and the projections.

Each choice of the M-chain gives slightly different, but algebraically equivalent, formulae (the assignments of α_κ, β_κ to $l_i l_i'$ and of m_a to M_k depend on the M-chain).

7. Concluding remarks

We have presented an interpretation of the combinatorial formula for the N-jm coefficients in terms of hidden angular momenta j_k and projections M_k. It is quite puzzling that the projections m_a of the N-jm coefficient appear only indirectly through the projections M_k as specified by

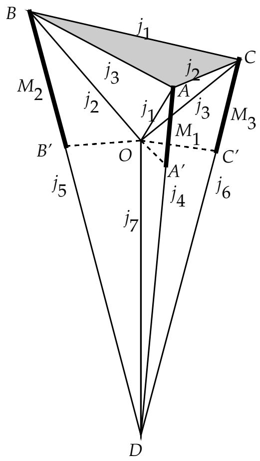


Figure 12. Representation of the 3-JM symbol in Euclidean geometry.

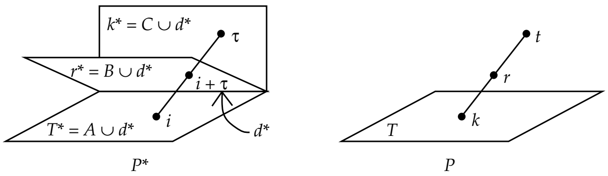


Figure 13. Partition $P^* = A \cup B \cup C \cup d^*$.

a M-chain. The comomenta, in the case of the $3n$-j coefficient, have been interpreted as *occupation numbers* in [15], but the physical interpretation of hidden angular momenta and projections is an open question.

We have drawn (figure 12) in three-dimensional Euclidean space the momenta and projections that take part in the construction of the projective space for the 3-j coefficient: length$[BD] = j_5$, length$[BB'] = M_2, \ldots$. The seven triangular conditions of P represented by collinearities in figure 8 now appear as triangles (triangle $j_1 j_2 j_3$ appears four times). When we take the centre t of P at j_7, the M-chain is $(1, 2, 3)$, the projections M_k are defined by equation (38) and the projections m_a by equation (39). The three points and three lines in the M-chain are pictured as the three edges and three faces adjacent to O in tetrahedron $OABC$. We finally consider a limit case in the spirit of Ponzano and Regge [19]. $OABC$ are kept fixed and D goes to infinity in the vertical direction. The projections M_k, m_k become genuine geometric projections of j_k on the vertical direction. In [19], this limit is used to obtain the 3-j coefficient, pictured by the shaded triangle ABC, from the 6-j coefficient, pictured by tetrahedron $ABCD$.

Appendix. Derivation of discrete Fourier transforms between momenta and comomenta

The dual hyperplanes in P^* of t, r and k are $T^* = t^* = A \cup d^*$, $r^* = B \cup d^*$ and $k^* = C \cup d^*$, where A, B, C, d^* is a partition of P^* and where d^* is the $(n-1)$-dimensional projective subspace dual to line tkr (see figure 13). Note that $\tau \in C$. From equation (27)

$$j_k = \tfrac{1}{2} \sum_{i \in A \cup B} l_i = \tfrac{1}{2} \sum_{i \in A} (l_i + l_{i+\tau}) = \tfrac{1}{2} \sum_{i \in T^* \setminus k^*} (l_i + l'_i) \tag{44}$$

which proves equation (32). Also from equation (27)

$$j_r = \tfrac{1}{2} \sum_{i \in A \cup C} l_i \qquad j_t = \tfrac{1}{2} \sum_{i \in B \cup C} l_i \tag{45}$$

so that

$$M_k = \tfrac{1}{2} \sum_{i \in A} l_i - \tfrac{1}{2} \sum_{i \in B} l_i = \tfrac{1}{2} \sum_{i \in A} (l_i - l_{i+\tau}) = \tfrac{1}{2} \sum_{i \in T^* \setminus k^*} (l_i - l'_i) \tag{46}$$

which proves equation (33).

The inverse transform, equations (34) and (35) results from properties of characters as in the case of the $3n$-j coefficient.

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